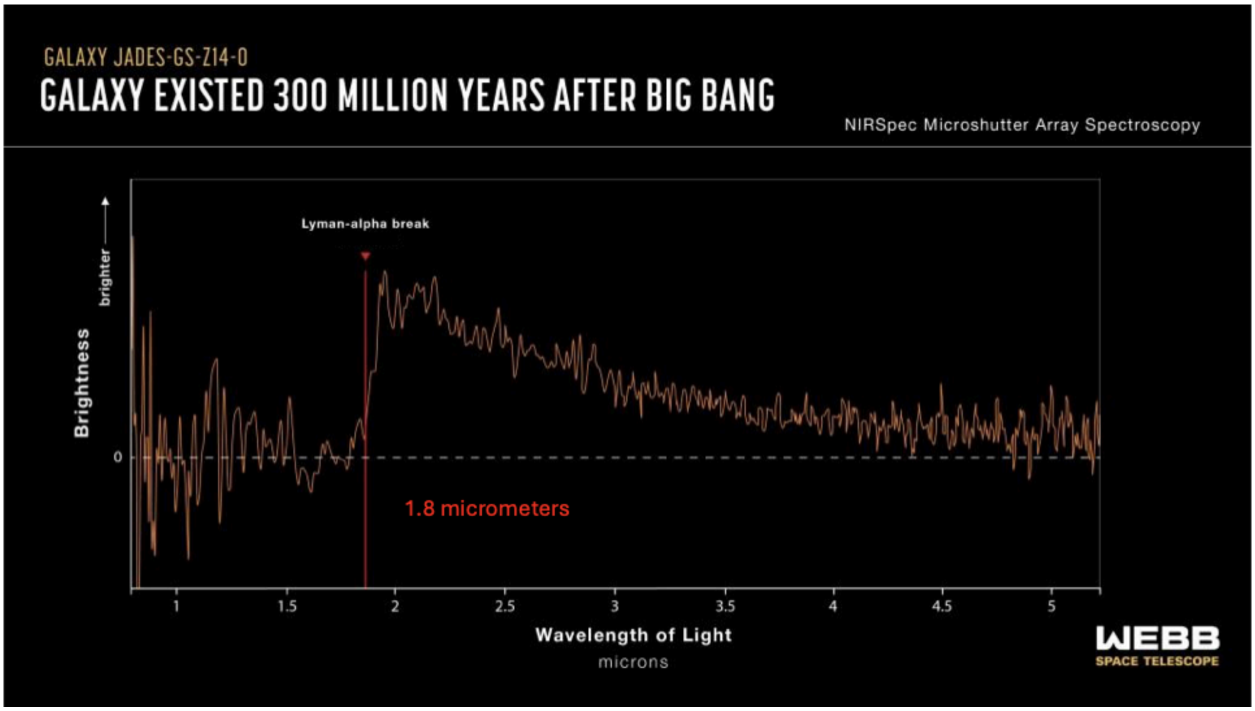
Problem Set 4: Galaxies, Relativity, and Cosmology [Answer Key]

Answer each question in 1 or 2 complete sentences, showing your work (math equations or illustrations) as needed.

1. The James Webb Space Telescope recently discovered the highest redshift galaxy ever! Below is the infrared (IR) spectrum:



   a. The Lyman-alpha break is a cutoff seen in galaxy spectra, indicated by the red line above. Below this wavelength, most photons are absorbed by hydrogen; in the rest frame, the Lyman-alpha break is at $\lambda_{emit} = 0.1216$ micrometers (microns). JWST detected it at $\lambda_{obs} = 1.8$ microns. What is the redshift of the galaxy? *(Hint: redshift is* $1 + z = \frac{\lambda_{obs}}{\lambda_{emit}}$*)*

Plugging in the observed and emitted wavelengths, we can solve for z:

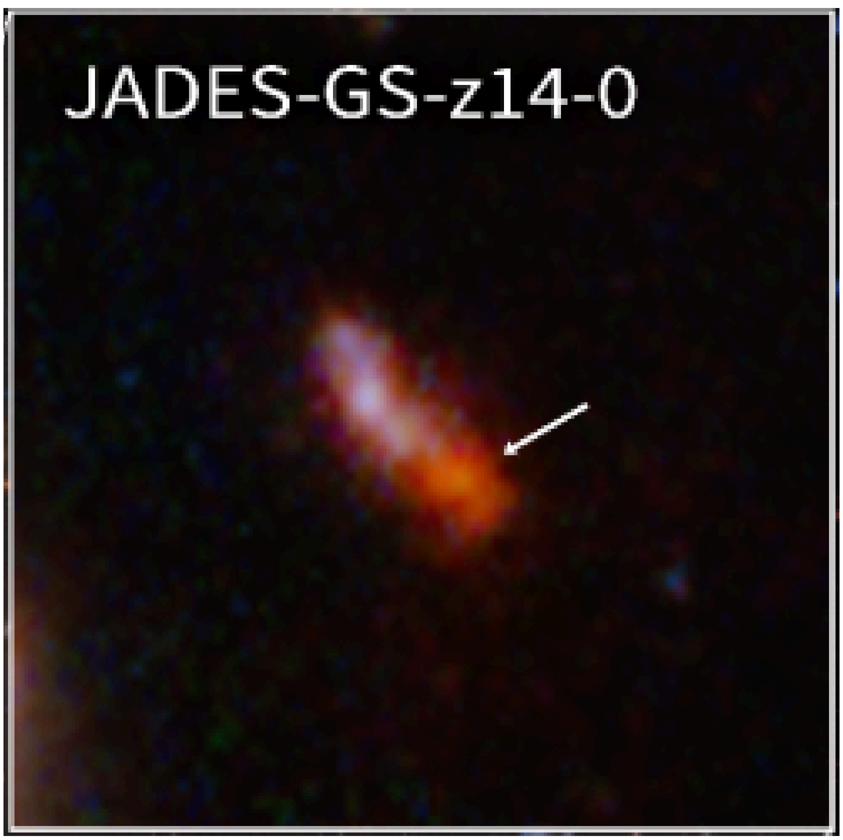
$z = \frac{1.8\ \mu m}{0.1216\ \mu m} - 1 = 13.8$

b. Using a Hubble constant $H_0 \approx 70\ km/s/Mpc$, what is this galaxy's distance in Megaparsec? (*Hint: remember Hubble's law,* $cz = H_0 D$)

We can rearrange Hubble's law and solve for the distance:

$$D = \frac{cz}{H_0} = \frac{(3\times10^5 km/s)\times13.8}{70\ km/s/Mpc} = 59,142\ Mpc$$

c. Here is an infrared image of the galaxy; using Hubble's classification scheme, is this a dwarf, elliptical, or spiral galaxy? If it is a spiral, is it a barred or normal galaxy?



This galaxy does not have a clear spiral, but is brighter towards the center compared to the halo, implying there may be a bar. While it is very faint given the distance, it must be extremely bright to be detectable at such a large distance;

therefore it's probably not a dwarf galaxy. This is most likely an elliptical or early type lenticular galaxy.

d. What kinds of stars do you think this galaxy contains: Population I, Population II, or Population III? Why?

This galaxy is at redshift z = 13.8, which means it was formed in the early universe, about 300 million years after the big bang. This is not long enough for Population II stars to have died and for Population I stars to form. This galaxy probably is composed of all Population II stars. Population III stars are theoretically formed from Hydrogen in the early universe; if these exist, they may also be found in this galaxy.

2. Believe it or not, the Christopher Nolan movie *Interstellar* is based on real and theoretical physics, including general relativity. Let's put it to the test! In one scene, the main characters visit a planet near a black hole and claim that, "One hour on this planet is equal to 7 years on Earth," because of the effects of gravitational time dilation.
    a. In units of the Schwarzschild radius ($r_s = \frac{2GM}{c^2}$) of the black hole, what is the distance to the black hole? (*Hint: for gravitational time dilation, the time that passes at an infinite distance from the black hole is* $\Delta t_{Earth} = \frac{\Delta t_{Planet}}{\sqrt{1 - \frac{2GM}{rc^2}}}$*)*

At first, it may seem like we need the Black Hole mass for this question, but we actually don't. Let the planet be located at a distance $r = Nr_s$ from the black hole, where N>1 so that it's outside of the event horizon. If we plug this in to the time dilation equation:

$$\Delta t_{Earth} = \frac{\Delta t_{Planet}}{\sqrt{1-\frac{2GM}{Nr_S c^2}}} = \frac{\Delta t_{Planet}}{\sqrt{1-\frac{1}{N}}}$$

Rearranging to solve for N:

$$N = \frac{1}{(\frac{\Delta t_{Planet}}{\Delta t_{Earth}})^2 - 1}$$

Finally, we plug in 7 years for the time on earth and 1 hour for the time on the planet:

$$N = \frac{1}{(\frac{7yr \times 365\ days/yr \times 24\ hr/day}{1\ hr})^2 - 1} = 2.7 \times 10^{-10}$$

So the planet has to be at exactly 2.7×10^{-10} times the Schwarzschild radius, which is way too close to escape the gravitational pull.

b. At one point, a character says they are trying to send data back to Earth, but it can't escape the black hole's gravitational pull. Why not?

Recall that if we are sending a signal in a strong gravitational field away from the massive object, the frequency will be lowered (the wavelength is increased) according to:

$$\frac{\nu_{obs}}{\nu_0} = \frac{\lambda_0}{\lambda_{obs}} = \sqrt{1 - \frac{2GM}{r_0 c^2}}$$

To transmit data, we need to send it on some carrier frequency, e.g. 1 GHz. But if we are at the event horizon so that $r_0 = r_S = \frac{2GM}{c^2}$, then the observed frequency will be:

$$\nu_{obs} = \nu_0 \sqrt{1 - \frac{2GMc^2}{2GMc^2}} = 0$$

So if we are very close to the event horizon, the light waves will be stretched so much that they will have a frequency of 0 (or an infinite wavelength) and will no longer be light waves. So they'll never escape the gravitational pull of the black hole to get back to earth.

c. [*Spoiler Alert!]* At the end of the movie, one of the astronauts returns home to find he now younger than his own daughter! Think about the twin "paradox" we discussed in class; does this make sense?

For this, we need to remember that the twin "paradox" is not actually a paradox! When our twins start their watches, they are in inertial reference frames, one that is at 'rest' on earth, the other moving close to the speed of light. The 'paradox' is that they both observe that the other's clock is running slow compared to their own. But this is not exactly a paradox because in order to move back to the rest frame, the moving twin slows down, so their frame is *accelerating*. Their frame is no longer inertial, and in fact, his clock will 'catch up' to his twin's during acceleration.

How does this relate to *Interstellar's* astronaut and his daughter? Rather than aging more slowly because he was moving close to light speed, the astronaut ages slowly because he is near a black hole – this is gravitational time dilation rather than special relativistic time dilation. This becomes an important difference; if the black hole were to suddenly disappear, the astronaut's clock would not 'catch up' to his daughter's clock like the twins' do. Instead, he remains at the age which he reached while in the gravitational field. In this case, *Interstellar's* ending makes sense!

d. ***Bonus:*** The waves on the planet are enormous – so large that they initially think the waves are mountains! Tidal waves on Earth are caused by the gravitational pull of the moon on the ocean; similarly, these mountain sized tidal waves are caused by the gravitational pull of the black hole. The maximum height of a wave is $h = \frac{3}{2}\frac{M_{BH}R_P^{\ 2}}{M_P D^3}$, where M_{BH} is the black hole mass, R_P is the planet's radius, M_P is the planet's mass, and D is the distance between the black hole and planet that you found in part a. If the wave they saw was the size of Mount Everest (8848 meters), and the planet is Earth-like ($M_P = 6 \times 10^{24} kg$ and $R_P = 6370\ km$) what is the mass of the black hole? (Newton's Gravitational Constant is $G = 6.67 \times 10^{-11} m^3/s^2/kg$)

First, let's plug in the Schwarzschild radius for $D = r_S = \frac{2GM_{BH}}{c^2}$ and simplify:

$$h = \frac{3}{2}\frac{M_{BH}R_P^2}{M_P(2GM_{BH}/c^2)^3} = \frac{3}{2}\frac{c^6R_P^2}{8M_PG^3M_{BH}^2}$$

Solving for the black hole mass:

$$M_{BH} = (\frac{3}{2}\frac{c^6R_P^2}{8M_PG^3})^{1/2} = (\frac{3}{16}\frac{(3\times10^8 m/s)^6(6370000m)^2}{(6\times10^{24}kg)(6.67\times10^{-11}m^3/s^2/kg)^3})^{1/2}$$

$$M_{BH} = 10^{31} kg = 13.5M_\odot$$

Or about 13.5 times the mass of the sun!